

Edge Covering Pseudo-outerplanar Graphs with Forests*

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Abstract

A graph is called pseudo-outerplanar if each block has an embedding on the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. In this paper, we prove that each pseudo-outerplanar graph admits edge decompositions into a linear forest and an outerplanar graph, or a star forest and an outerplanar graph, or two forests and a matching, or $\max\{\Delta(G), 4\}$ matchings, or $\max\{\lceil \Delta(G)/2 \rceil, 3\}$ linear forests. These results generalize some ones on outerplanar graphs and $K_{2,3}$ -minor-free graphs, since the class of pseudo-outerplanar graphs is a larger class than the one of $K_{2,3}$ -minor-free graphs.

Keywords: pseudo-outerplanar graphs; edge decomposition; edge chromatic number; linear arboricity.

1 Introduction

In this paper, all graphs considered are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G , respectively. Let $d_G(v)$ (or $d(v)$ for simplicity) denote the *degree* of a vertex $v \in V(G)$. A *block* is a maximal 2-connected subgraph of a given graph G . A graph H is a *minor* of a graph G if a copy of H can be obtained from G via repeated edge deletion and/or edge contraction. For a subset $S \subseteq V(G) \cup E(G)$, $G[S]$ denotes the subgraph of G induced by S . The *vertex connectivity* of a graph G , denoted by $\kappa(G)$, is the minimum number of vertices whose deletion from G disconnects it. For other undefined concepts we refer the readers to [3].

An *outerplanar graph* is a graph that can be embedded on the plane in such a way that it has no crossings and that all its vertices lie on the outer face. In this paper, we aim to introduce an extension of this concept. A graph is called *pseudo-outerplanar* if each block has an embedding on the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. In this embedding, the edges bounding the disk(s) are called *boundary edges* and a disk is said to be *closed* or *open* according to whether or not it contains the circle that constitutes its boundary. For example, Figure 1 exhibits a pseudo-outerplanar embedding of a graph with two blocks: one is K_4 and the other is $K_{2,3}$. The drawing of K_4 in this embedding lies inside a closed disk but the one of $K_{2,3}$ in this embedding lies inside an open disk. In Figure 1, the edges in bold are the boundary edges. A pseudo-outerplanar graph is *maximal* if it is not possible to add an edge such that the resulting graph is still pseudo-outerplanar. Thus $K_{2,3}$ is not a maximal pseudo-outerplanar graph, since we can possibly add two edges to $K_{2,3}$ and remain its pseudo-outerplanarity. One can easily check that each pseudo-outerplanar graph has a planar embedding by its definition. So the class of pseudo-outerplanar graphs forms a subclass of planar graphs. Actually, the definition of pseudo-outerplanar graphs are similar to that of 1-planar graphs (i.e. graphs that can be drawn on the plane so that each edge is crossed by at most one other edge), which was introduced by Ringel [10].

Many classic problems in graph theory are considered for the class of planar graphs and its subclasses, such as the class of series-parallel graphs and the one of outerplanar graphs. Taking the problem of covering graphs with

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*Research supported by NSFC (10971121, 11101243, 61070230), RFDP (20100131120017) and GII FSDU (yzc10040).

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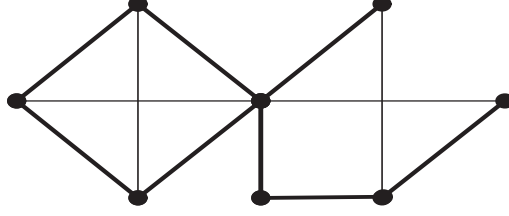


Figure 1: An example of pseudo-outerplanar

forests and a graph of bounded maximum degree for example, we say that a graph is (t, d) -coverable if its edges can be covered by at most t forests and a graph of maximum degree d . In [2], Balogh et al. conjectured that every simple planar graph is $(2, 4)$ -coverable and gave an example to show that there are infinitely many planar graphs that are not $(2, 3)$ -coverable. This conjecture was recently confirmed by Gonçalves in [5]. In [2], it is also proved that every series-parallel graph is $(2, 0)$ -coverable and that every $K_{2,3}$ -minor-free graph is both $(1, 3)$ -coverable and $(2, 0)$ -coverable. Since a graph is outerplanar if and only if it is $\{K_4, K_{2,3}\}$ -minor-free [8], every outerplanar graph is both $(1, 3)$ -coverable and $(2, 0)$ -coverable. It is interesting to know what can be said about pseudo-outerplanar graphs, another larger class than outerplanar graphs.

Edge-coloring is another classic problem in graph theory. In fact, we can regard edge-coloring problems as a covering problem. When we color the edges of a graph G , our actual task is to decompose the edge set $E(G)$ into some parts such that the graph induced by each part satisfies a property \mathcal{P} . Different properties \mathcal{P} correspond to different types of edge-coloring. For example, a *proper k -edge-coloring* of G is a decomposition of $E(G)$ into k subsets such that the graph induced by each subset is a matching in G . The minimum integer k such that G has a proper k -edge-coloring, denoted by $\chi'(G)$, is the *edge chromatic number* of G . Vizing's Theorem states that for any graph G , $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. A graph G is said to be of *class 1* if $\chi'(G) = \Delta(G)$, and of *class 2* if $\chi'(G) = \Delta(G) + 1$. To determine whether a planar graph is of class 1 is an interesting problem. Sanders and Zhao [11] showed that each planar graph with maximum degree at least 7 is of class 1. Juvan, Mohar and Thomas [9] proved that each series-parallel graph with maximum degree at least 3 is of class 1, and thus holds for outerplanar graphs. It is open whether each pseudo-outerplanar graph with large maximum degree is of class 1.

On the other hand, one can consider improper edge-colorings. Concerning this topic, Harary [7] introduced the concept of linear arboricity. A *linear forest* is a forest in which every connected component is a path. A *k -tree-coloring* of G is a decomposition of $E(G)$ into k subsets such that the graph induced by each subset is a linear forest. The *linear arboricity* $la(G)$ of a graph G is the minimum integer k such that G has a k -tree-coloring. Akiyama, Exoo and Harary [1] conjectured that $la(G) = \lceil (\Delta(G) + 1)/2 \rceil$ for any regular graph G . It is obvious that $la(G) \geq \lceil \Delta(G)/2 \rceil$ for any graph G and $la(G) \geq \lceil (\Delta(G) + 1)/2 \rceil$ for any regular graph G . Hence the conjecture is equivalent to the following one.

Conjecture 1.1 (Linear Arboricity Conjecture). *For any graph G , $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.*

Now Conjecture 1.1 has been proved true for all planar graphs (see [13, 15]). However, it is still interesting to determine which kinds of planar graphs satisfy $la(G) = \lceil \Delta(G)/2 \rceil$. Wu [13] proved that it holds for planar graphs with maximum degree at least 13. And the bound 13 was later improved to 9 by Cygan et al. [4]. For subclasses of planar graphs, Wu [14] proved that $la(G) = \lceil \Delta(G)/2 \rceil$ for all series-parallel graphs (hence also for all outerplanar graphs) with maximum degree at least 3. Can the same conclusion extend to the class of pseudo-outerplanar graphs?

In Section 2, we give some relationships among three classes containing the outerplanar graphs; they are the $K_{2,3}$ -minor-free graphs, the series-parallel graphs and the pseudo-outerplanar graphs. In Section 3, we investigate the problem of covering pseudo-outerplanar graphs with forests and a graph of bounded maximum degree. In Section 4, some unavoidable structures of pseudo-outerplanar graphs are obtained. These structures will be applied

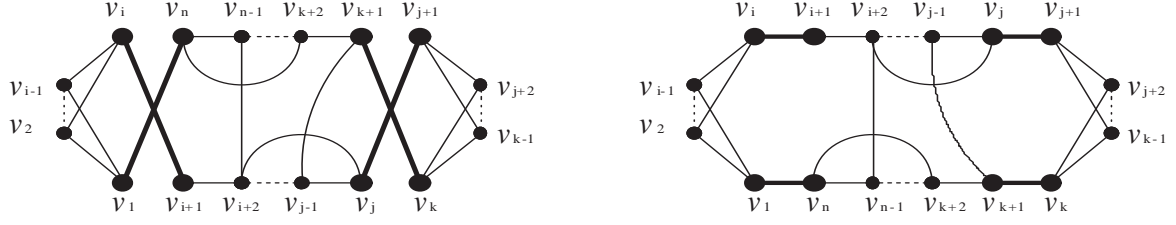


Figure 2: Each hamiltonian pseudo-outerplanar graphs has a hamiltonian diagram

to determine the edge chromatic number and linear arboricity of pseudo-outerplanar graphs in Section 5.

2 Basic Properties

Let G be a pseudo-outerplanar graph. In the following of this paper, we always assume that G has been drawn on the plane such that (1) for each block B of G , the vertices of B lie on a fixed circle and the edges of B lie inside the disk of this circle with each of them crossing at most one another; (2) the number of crossings in G is as small as possible. This drawing is called a *pseudo-outerplanar diagram* of G . Let G be a pseudo-outerplanar diagram and let B be a block of G . Denote by $v_1, v_2, \dots, v_{|B|}$ the vertices of B , which are lying in a clockwise sequence. Let $\mathcal{V}[v_i, v_j] = \{v_i, v_{i+1}, \dots, v_j\}$ and $\mathcal{V}(v_i, v_j) = \mathcal{V}[v_i, v_j] \setminus \{v_i, v_j\}$, where the subscripts and the additions are taken modular $|B|$.

Lemma 2.1. [8] *Let G be an outerplanar graph. Then*

- (a) $\delta(G) \leq 2$,
- (b) $\kappa(G) \leq 2$.

Theorem 2.2. *Let G be a pseudo-outerplanar graph. Then*

- (a) $\delta(G) \leq 3$,
- (b) $\kappa(G) \leq 2$ unless $G \simeq K_4$.

Proof. The proof of (a) is left to Corollary 4.3. So we only prove (b) here. If $|G| \leq 4$, then this theorem is trivial. So we assume that G is a pseudo-outerplanar diagram with $|G| \geq 5$ and $\kappa(G) \geq 3$. If G has no crossings, then G is an outerplanar graph and thus by Lemma 2.1, $\kappa(G) \leq 2$, a contradiction. So we assume that there are two chords $v_i v_j$ and $v_k v_l$ in G that cross each other, and that v_i, v_k, v_j, v_l are lying in a clockwise sequence. Since $|G| \geq 5$, at least one of $\mathcal{V}(v_i, v_k)$, $\mathcal{V}(v_k, v_j)$, $\mathcal{V}(v_j, v_l)$ and $\mathcal{V}(v_l, v_i)$ is nonempty. Without loss of generality, assume that $\mathcal{V}(v_i, v_k) \neq \emptyset$. Since $v_i v_j$ crosses $v_k v_l$, there is no edges between the two vertex sets $\mathcal{V}(v_i, v_k)$ and $\mathcal{V}(v_k, v_i)$. So $\{v_i, v_k\}$ separates $\mathcal{V}(v_i, v_k)$ and $\mathcal{V}(v_k, v_i)$, contradicting to $\kappa(G) \geq 3$. \square

It is well-known that every 2-connected outerplanar graph is hamiltonian. But this result does not hold for 2-connected pseudo-outerplanar graphs. The complete bipartite graph $K_{2,3}$ is such a counterexample. A 2-connected pseudo-outerplanar diagram is called a *hamiltonian diagram* if it is in such a way that all its vertices lie on a *closed circuit* C (i.e. the disk of C is closed). This closed circuit C is called the *hamiltonian boundary* of the diagram. By this definition, one can easily see that a non-hamiltonian 2-connected pseudo-outerplanar graph cannot have a hamiltonian diagram. It seems interesting to answer whether each hamiltonian pseudo-outerplanar graph has a hamiltonian diagram.

Theorem 2.3. *Let G be a pseudo-outerplanar diagram and C be a hamiltonian cycle of G . If C is not the boundary of G , then G has a hamiltonian diagram such that C is the hamiltonian boundary of this diagram.*

Proof. We proceed by induction on the order of G . Since G has a hamiltonian cycle $C = v_1v_2 \cdots v_nv_1$ that is not the boundary of the pseudo-outerplanar diagram of G , one can easily deduce that there exists at least one crossing in the drawing of C (a sub-diagram of G indeed). Suppose that v_jv_{j+1} and v_kv_{k+1} ($j < k$) cross each other and that v_j follows v_k in a clockwise walk around G . Denote respectively by U and W the set of vertices from v_j to v_{k+1} and from v_{j+1} to v_k in the cyclic clockwise sequence of vertices on the outer boundary of G . Take the first graph in Figure 2 for example, we have $C = v_1v_2 \cdots v_nv_1$, $U = \{v_j, v_{j-1}, \dots, v_{i+1}, v_1, \dots, v_i, v_n, v_{n-1}, \dots, v_{k+1}\}$ and $W = \{v_{j+1}, v_{j+2}, \dots, v_{k-1}, v_k\}$. Note that besides v_jv_{j+1} and v_kv_{k+1} , there is no other edge uw such that $u \in U$ and $w \in W$ by the definition of G . Let $G_1 = G[U] + v_jv_{k+1}$ and $G_2 = G[W] + v_{j+1}v_k$. Then G_1 is a pseudo-outerplanar diagram with a hamiltonian cycle $C_1 = v_{k+1}v_{k+2} \cdots v_nv_1 \cdots v_jv_{k+1}$ while G_2 is a pseudo-outerplanar diagram with a hamiltonian cycle $C_2 = v_{j+1}v_{j+2} \cdots v_kv_{j+1}$. By induction hypothesis, G_1, G_2 respectively has a hamiltonian diagram such that C_1, C_2 is the hamiltonian boundary of each diagram. Now we combine these two hamiltonian diagrams and add two edges v_jv_{j+1} and v_kv_{k+1} (see the second graph in Fig.2), then we can get a hamiltonian diagram of G with hamiltonian boundary $v_{k+1}v_{k+2} \cdots v_nv_1 \cdots v_jv_{j+1}v_{j+2} \cdots v_{k-1}v_kv_{k+1}$, which is the cycle C indeed. \square

Corollary 2.4. *Each hamiltonian pseudo-outerplanar graph has a hamiltonian diagram.*

We say a graph G *quasi-hamiltonian* if each block of G is hamiltonian. Denote the class of pseudo-outerplanar graphs, quasi-hamiltonian pseudo-outerplanar graphs, series-parallel graphs, $K_{2,3}$ -minor-free graphs and outerplanar graphs by \mathcal{P} , \mathcal{P}_H , \mathcal{S} , $\mathcal{M}_{2,3}$ and \mathcal{O} , respectively. The following basic relationship is obvious.

Remark 2.5. $\mathcal{P} \supset \mathcal{P}_H \supset \mathcal{O}$, $\mathcal{M}_{2,3} \cap \mathcal{S} = \mathcal{O}$

In the following, we continue to study some more interesting relationships among these five classes of graphs.

Theorem 2.6. $\mathcal{P}_H \cap \mathcal{S} = \mathcal{O}$.

Proof. Let $G \in \mathcal{P}_H \cap \mathcal{S}$ and let B be a block of G . By Corollary 2.4 B has a hamiltonian diagram, and actually this diagram is outerplanar. If there was a crossing, there would be four vertices u, v, x, y with uv and xy crossing in B . Since the diagram is hamiltonian, there are four pairwise disjoint paths P_{ux}, P_{xv}, P_{vy} and P_{yu} that connects u to x , x to v , v to y and y to u . Thus the edges uv and vy and the four paths $P_{ux}, P_{xv}, P_{vy}, P_{yu}$ form a K_4 -minor, which is impossible in a series-parallel graph. Hence B is an outerplanar graph. \square

Lemma 2.7. [6] *Let H be a graph obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2 and let G be a H -minor-free graph. Then each block of G is either K_4 -minor-free or isomorphic to K_4 .*

Corollary 2.8. *For any 2-connected graph $G \in \mathcal{M}_{2,3}$, either $G \in \mathcal{O}$ or $G \simeq K_4$.*

Proof. Since $G \in \mathcal{M}_{2,3}$, G is H -minor-free where H is the graph in Lemma 2.7. Thus by Remark 2.5 and Lemma 2.7 either $G \in \mathcal{O}$ or $G \simeq K_4$. \square

Theorem 2.9. $\mathcal{M}_{2,3} \subset \mathcal{P}_H$.

Proof. The inclusion of $\mathcal{M}_{2,3}$ in \mathcal{P}_H directly follows from Corollary 2.8. The inequality comes from the graph $(K_1 \cup K_2) \vee \overline{K_2}$ that belongs to \mathcal{P}_H but not to $\mathcal{M}_{2,3}$. \square

3 Decomposability

Let G be a pseudo-outerplanar diagram and let B be a block of G . Denote by $v_1, v_2, \dots, v_{|B|}$ the vertices of B , which are lying in a clockwise sequence. The edges of the form v_iv_j ($j - i = 1$ or $|B| - 1$) are called *boundaries* while the edges of the form v_iv_j ($1 < j - i < |B| - 1$) are called *chords* of G . Since G is a pseudo-outerplanar diagram, all the crossings are generated by one chord crossing another chord. Let $\mathcal{C}[v_i, v_j]$ be the set of chords xy with $x, y \in \mathcal{V}[v_i, v_j]$ and let $\mathcal{C}(G)$ be the set of crossed chords in G .

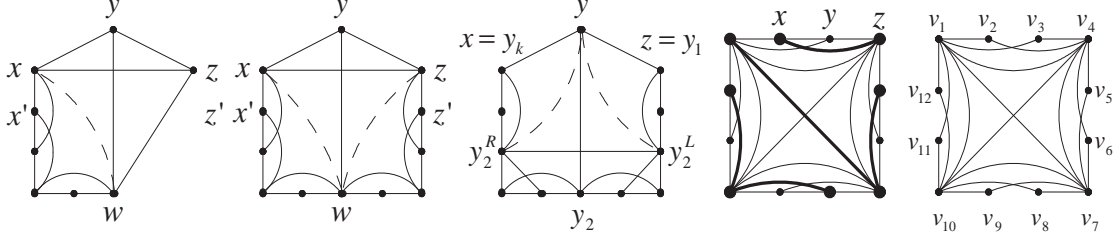


Figure 3: Decomposability of pseudo-outerplanar graphs

Theorem 3.1. *Let G be a hamiltonian pseudo-outerplanar diagram and C be the hamiltonian boundary of this diagram. Let $y \in V(C)$ and $yx, yz \in E(C)$. Then there exists a linear forest T in G such that $E(T) \subseteq \mathcal{C}(G)$, $d_T(y) = 0$, $\max\{d_T(x), d_T(z)\} \leq 1$, and $G - E(T)$ is an outerplanar diagram.*

Proof. We proceed by induction on the order of G . One can see that the theorem holds for $|G| \leq 4$ since the case $G = K_4$ is trivial. So we assume that $|G| \geq 5$. In the following, we also assume that the three vertices x, y, z occur on C in a clockwise sequence.

First, we consider the case when $d_G(y) = 2$. Let $G' = (G - y) + xz$ and $C' = (C - y) + xz$ (note that if the edge xz already exists in G , we let $G' = G - y$ and $C' = C - y$). Then G' is a hamiltonian pseudo-outerplanar diagram with C' being its hamiltonian boundary. Let $xx' \in E(C')$ with $x' \neq z$ (x' exists because $|V(G)| \geq 5$). By induction on (G', C', x', x, z) (as (G, C, x, y, z) , respectively), there exists a linear forest T' in G' such that $E(T') \subseteq \mathcal{C}(G')$, $d_{T'}(x) = 0$, $\max\{d_{T'}(x'), d_{T'}(z)\} \leq 1$, and $G' - E(T')$ is an outerplanar diagram. Note that $\mathcal{C}(G') = \mathcal{C}(G)$. Let $T = T'$. Then $E(T) \subseteq \mathcal{C}(G)$, $d_T(x) = d_T(y) = 0$ and $d_T(z) \leq 1$. Furthermore, one can easily see that $G - E(T)$ is an outerplanar diagram.

If $d_G(y) = 3$ and $xz \in E(G)$, then the edge xz is crossed by another edge yw . Assume first that $\mathcal{V}(z, w) = \emptyset$, then $zw \in E(C)$. Let $G' = G[\mathcal{V}[w, x]] + wx$ and let C' be the cycle consisting of the edge xw and the clockwise subpath around C from w to x . We assume that $N_{C'}(x) \setminus \{w\} \neq \emptyset$, because otherwise G would have less than five vertices, a contradiction. Let $xx' \in E(C')$ with $x' \neq w$ (see 1st graph of Figure 3). Note that G' is a hamiltonian pseudo-outerplanar diagram with C' being its hamiltonian boundary. By induction on (G', C', x', x, w) , there exists a linear forest T' in G' such that $E(T') \subseteq \mathcal{C}(G')$, $d_{T'}(x) = 0$, $\max\{d_{T'}(x'), d_{T'}(w)\} \leq 1$, and $G' - E(T')$ is an outerplanar diagram. Let $T = T' + xz$. Then $E(T) \subseteq \mathcal{C}(G)$, $d_T(y) = 0$, $d_T(x) = d_T(z) = 1$, and $G - E(T)$ is an outerplanar diagram. Thus a linear forest T as required has been constructed. So in the following, we assume that $\mathcal{V}(z, w) \neq \emptyset$ and $\mathcal{V}(w, x) \neq \emptyset$. Let $zz' \in E(C_1)$ with $z' \neq y, w$, and let $xx' \in E(C)$ with $x' \neq y, w$ (see 2nd graph of Figure 3). Set $G_1 = G[\mathcal{V}[z, w]] + zw$ and $G_2 = G[\mathcal{V}[w, x]] + wx$. By C_1 and C_2 , we respectively denote the cycle that consists of the edge wz and the clockwise subpath around C from z to w , and that consists of the edge xw and the clockwise subpath around C from w to x . Then for $i = 1, 2$, G_i is a hamiltonian pseudo-outerplanar diagram with C_i being its hamiltonian boundary. By inductions on (G_1, C_1, w, z, z') and (G_2, C_2, w, x, x') , there respectively exists a linear forest T_1 in G_1 with $E(T_1) \subseteq \mathcal{C}(G_1)$, $d_{T_1}(z) = 0$, $\max\{d_{T_1}(w), d_{T_1}(z')\} \leq 1$ and $G_1 - E(T_1)$ being an outerplanar diagram, and a linear forest T_2 in G_2 with $E(T_2) \subseteq \mathcal{C}(G_2)$, $d_{T_2}(x) = 0$, $\max\{d_{T_2}(w), d_{T_2}(x')\} \leq 1$ and $G_2 - E(T_2)$ being an outerplanar diagram. Let $T = T_1 \cup T_2 \cup \{xz\}$. Then we can easily see that $E(T) \subseteq \mathcal{C}(G)$, $d_T(y) = 0$, $d_T(x) = d_T(z) = 1$, $d_T(w) \leq 2$ and $G - E(T)$ is an outerplanar diagram. Since T_1 and T_2 intersect on at most one vertex, w , of degree at most one in each forest and there is no edges between $V(T_1) \setminus \{w\}$ and $V(T_2) \setminus \{w\}$, $T_1 \cup T_2$ is a linear forest. Furthermore since x, y and z have degree 0 in $T_1 \cup T_2$, $T_1 \cup T_2 \cup \{xz\}$ is as required.

The last case is when $d_G(y) \geq 3$ and $xz \notin E(G)$. We label the neighbors of y by y_1, y_2, \dots, y_k in a clockwise sequence on C , where $y_1 = z$, $y_k = x$ and $k \geq 3$. If yy_2 is not a crossed chord in G , then set $G_1 = G[\mathcal{V}[y, y_2]]$ and $G_2 = G[\mathcal{V}[y_2, y]]$. Denote by C_1 (resp. C_2) the cycle consisting of the edge yy_2 and the clockwise subpath around C from y to y_2 (resp. from y_2 to y). Then G_i ($i = 1, 2$) is a hamiltonian pseudo-outerplanar diagram with C_i being

its hamiltonian boundary. By using inductions on (G_1, C_1, y_2, y, z) and (G_2, C_2, y_2, y, x) , it is easy to construct a linear forest as required. So we assume that yy_2 is crossed by another edge $y_2^L y_2^R$ in G , where y_2^L, y_2, y_2^R are labeled clockwise. Since there is no edges between $\mathcal{V}(y, y_2^L)$ and $\mathcal{V}(y_2^L, y)$, or between $\mathcal{V}(y, y_2^R)$ and $\mathcal{V}(y_2^R, y)$, we can add two edges yy_2^L and yy_2^R to G if they do not really exist so that they do not generate new crossings in G and thus the resulting graph is still pseudo-outerplanar (see the 3rd graph of Fig. 3). By C_1, C_2 and C_3 , we respectively denote the cycle that consists of the edge $y_2^L y$ and the clockwise subpath around C from y to y_2^L , and that consists of the path $y_2^R y y_2^L$ and the clockwise subpath around C from y_2^L to y_2^R , and that consists of the edge yy_2^R and the clockwise subpath around C from y_2^R to y . Let G_i be the subgraph of G contained in the closed disc of C_i ($i = 1, 2, 3$). Here one should be careful that if $y_2^L = y_1$ (resp. $y_2^R = y_k$), then C_1 (resp. C_3) is not a cycle indeed and then G_1 (resp. G_3) is defined to be a null graph. However, G_1 and G_3 cannot simultaneously be null graphs, since $y_1 y_k \notin E(G)$. Hence any of G_i ($i = 1, 2, 3$) is a subgraph of G with smaller order. Moreover, every non-null graph G_i is a hamiltonian pseudo-outerplanar diagram with C_i being its hamiltonian boundary. Without loss of generality, we assume that none of G_i ($i = 1, 2, 3$) is null graph. By inductions on $(G_1, C_1, y_1, y, y_2^L)$, $(G_2, C_2, y_2^R, y, y_2^L)$ and $(G_3, C_3, y_k, y, y_2^R)$, there exists a linear forest T_i in G_i such that $E(T_i) \in \mathcal{C}(G_i)$, $d_{T_i}(y) = 0$ and $G_i - E(T_i)$ is an outerplanar diagram ($i = 1, 2, 3$). Meanwhile, we have $\max\{d_{T_1}(y_1), d_{T_1}(y_2^L), d_{T_2}(y_2^L), d_{T_2}(y_2^R), d_{T_3}(y_2^R), d_{T_3}(y_k)\} \leq 1$. Let $T = T_1 \cup T_2 \cup T_3$. Note that there is no edges whose end points are belong to different parts of the vertex partition $[\mathcal{V}(y, y_2^L), \mathcal{V}(y_2^L, y_2^R), \mathcal{V}(y_2^R, y)]$ (because otherwise either yy_2 or $y_2^L y_2^R$ may be crossed twice). So T is still a forest. Since $d_T(y_2^R) \leq d_{T_2}(y_2^R) + d_{T_3}(y_2^R) \leq 2$ and $d_T(y_2^L) \leq d_{T_1}(y_2^L) + d_{T_2}(y_2^L) \leq 2$, $\Delta(T) \leq 2$. Thus, a linear forest T has been constructed. Since $\mathcal{C}(G_i) \subseteq \mathcal{C}(G)$ ($i = 1, 2, 3$), $E(T) = E(T_1) \cup E(T_2) \cup E(T_3) \in \mathcal{C}(G_1) \cup \mathcal{C}(G_1) \cup \mathcal{C}(G_3) \in \mathcal{C}(G)$. Meanwhile, $d_T(y) = d_{T_1}(y) + d_{T_2}(y) + d_{T_3}(y) = 0$, $d_T(x) = d_T(y_k) = d_{T_3}(y_k) \leq 1$ and $d_T(z) = d_T(y_1) = d_{T_1}(y_1) \leq 1$. At last since $G - E(T) \subseteq \bigcup_{i=1}^3 (G_i - E(T_i))$, $G - E(T)$ is an outerplanar diagram. Hence we construct a linear forest T as required in G and completes the proof of the theorem. \square

A *star forest* is a graph in which every component is a star. The *root* of a star is the vertex of maximum degree. Note that K_2 has two roots. The *roots* of a star forest is the union of the root of each star component. The following Theorem 3.2 is an analog of Theorem 3.1 (note that the condition $\max\{d_T(x), d_T(z)\} \leq 1$ in Theorem 3.1 is equivalent to that x or z are vertices of T if and only if x or z are leaves of T), whose proof is almost the same with that of Theorem 3.1. Actually, we can still proceed by induction on the order of G and split the proofs into three cases: the first is $d_G(y) = 2$, the second is $d_G(y) = 3$ and $xz \in E(G)$, and the last is $d_G(y) \geq 3$ and $xz \notin E(G)$. In each case we can construct a star forest T as required by the same way as in the proof of Theorem 3.1. The detailed proof of Theorem 3.2 is left to the readers.

Theorem 3.2. *Let G be a hamiltonian pseudo-outerplanar diagram and C be the hamiltonian boundary of this diagram. Let $y \in V(C)$ and $yx, yz \in E(C)$. Then there exists a star forest T in G such that $E(T) \in \mathcal{C}(G)$, $d_T(y) = 0$, x or z are vertices of T if and only if x or z are roots of T , and $G - E(T)$ is an outerplanar diagram.*

Corollary 3.3. *Each pseudo-outerplanar graph can be decomposed into an outerplanar graph and a linear forest, or an outerplanar graph and a star forest.*

Proof. Without loss of generality, let G be a quasi-hamiltonian pseudo-outerplanar diagram. Otherwise we can add some edges to close the circumferential boundary of each block. In what follows, we proceed by induction on the number of blocks, $\omega(G)$, in G . The base case when $\omega(G) = 1$ follows directly from Theorems 3.1 and 3.2 so we assume that $\omega(G) \geq 2$. Choose a block B of G that contains only one cut vertex y (i.e. B is an end-block). By Theorems 3.1 and 3.2, B can be decomposed into an outerplanar graph H_1 and a linear forest T_1 with $d_{T_1}(y) = 0$, or an outerplanar graph H_2 and a star forest T_2 with $d_{T_2}(y) = 0$. Meanwhile, by the induction hypothesis, $G - B$ can also be decomposed into an outerplanar graph H_3 and a linear forest T_3 , or an outerplanar graph H_4 and a star forest T_4 . Therefore, G can be covered by the linear forest $T = T_1 \cup T_3$ and the outerplanar graph $H = H_1 \cup H_3$, or the star forest $T = T_2 \cup T_4$ and the outerplanar graph $H = H_2 \cup H_4$. \square

Theorem 3.4. *For every integer $n \geq 12$, there exists a 2-connected pseudo-outerplanar graph with order n that cannot be decomposed into an outerplanar graph and a matching.*

Proof. We show the last graph G in Figure 3 is a graph that cannot be decomposed into an outerplanar graph and a matching. Otherwise we suppose that $E(G) = E(H) \cup E(M)$, where H is an outerplanar and M is matching. Set $S_i = \{v_i v_{i+1}, v_i v_{i+2}, v_i v_{i+3}, v_{i+1} v_{i+3}, v_{i+2} v_{i+3}\} \pmod{12}$ ($i = 1, 4, 7, 10$). We then claim that there exists an edge set S_i that is contained in $E(H)$. Suppose not, assume first that $v_1 v_2 \in E(M)$. Then $v_1 v_k \in E(H)$ ($k = 3, 4, 7, 10, 11, 12$) and exactly one of $v_{10} v_{11}$ and $v_{10} v_{12}$ should be contained in $E(M)$, say $v_{10} v_{11}$. Then $v_k v_{10} \in E(H)$ ($k = 4, 7, 12$). However, the five vertices $\{v_1, v_4, v_7, v_{10}, v_{12}\}$ and the three disjoint paths $\{v_1 v_4 v_{10}, v_1 v_7 v_{10}, v_1 v_{12} v_{10}\}$ form a copy of $K_{2,3}$ in H ; this is a contradiction. Now assume that $v_1 v_4 \in E(M)$. Then $v_1 v_2, v_1 v_3, v_1 v_7, v_2 v_4, v_3 v_4, v_4 v_7 \in E(H)$ and thus the graph induced by $\{v_1, v_2, v_3, v_4, v_7\}$ is a $K_{2,3}$, which is impossible in an outerplanar graph. Hence in the following we assume that $S_1 \subseteq E(H)$. If $\{v_1 v_7, v_4 v_7\} \subseteq E(H)$, then the five vertices $\{v_1, v_2, v_3, v_4, v_7\}$ and the three disjoint paths $\{v_1 v_2 v_4, v_1 v_3 v_4, v_1 v_7 v_4\}$ form a copy of $K_{2,3}$ in H , a contradiction. So exactly one of $v_1 v_7$ and $v_4 v_7$ should be contained in $E(M)$, say $v_1 v_7$. Similarly, $\{v_1 v_{10}, v_4 v_{10}\} \not\subseteq E(H)$. Thus $v_1 v_{10} \in E(H)$, $v_4 v_{10} \in E(M)$ and $v_7 v_{10} \in E(H)$. Now the six vertices $\{v_1, v_2, v_3, v_4, v_7, v_{10}\}$ and the three disjoint paths $\{v_1 v_3 v_4, v_1 v_2 v_4, v_1 v_{10} v_7 v_4\}$ form a $K_{2,3}$ -minor in H . This contradiction completes the proof of this theorem. \square

Theorem 3.5. *Every maximal pseudo-outerplanar graph G is obtained from a maximal pseudo-outerplanar diagram H by gluing a K_3 or a K_4 along a boundary edge of H .*

Proof. Without loss of generality, we assume that G is a 2-connected maximal pseudo-outerplanar diagram. Since G is maximal, G is hamiltonian and G has at least one chord. Let $C = \{v_1 v_2 \cdots v_{|G|}\}$ be the hamiltonian boundary of the diagram of G . Now we split the proof into two cases.

Case 1. There exists a crossed chord in G .

Let $v_i v_j$ be a chord in G that crosses another chord $v_k v_l$ ($1 \leq i < k < j < l \leq |G|$). Actually, we can properly choose i and j such that there is no pair of mutually crossed chords in $\mathcal{C}[v_i, v_l] \setminus \{v_i v_j, v_k v_l\}$, because otherwise we can change the value of i or j to meet this condition.

Assume first that there is no non-crossed chord in $\mathcal{C}[v_i, v_l] \setminus \{v_i v_l\}$. Then we shall have $k = i + 1$. Otherwise, since $v_i v_k \notin E(G)$ by our assumption, we can add $v_i v_k$ to G so that G is still pseudo-outerplanar, contradicting the fact that G is maximal. Similarly, $j = k + 1$, $l = j + 1$ and $v_i v_l \in E(G)$ by the maximality of G . Furthermore, $d(v_k) = d(v_j) = 3$. Now remove the vertices v_k and v_j from G and denote the resulting graph by H . Then H is a maximal pseudo-outerplanar diagram. Otherwise we can add an edge $e = v_a v_b \notin E(H)$ ($a, b \neq k$ or j) to H so that $H + e$ is pseudo-outerplanar. Therefore, $e \notin E(G)$ and $G + e$ is a pseudo-outerplanar graph, contradicting the fact that G is maximal. At this stage, one can easily see that G is obtained from H by gluing a K_4 along the boundary edge $v_i v_l$ of H .

Second, assume that there is a non-crossed chord $v_r v_s$ in $\mathcal{C}[v_i, v_l] \setminus \{v_i v_l\}$. Since there is no crossed chords in $\mathcal{C}[v_r, v_s]$ by assumption, we can properly choose r and s such that $\mathcal{C}[v_r, v_s] \setminus \{v_r v_s\} = \emptyset$. By the maximality of G , we have $s = r + 2$, otherwise we can add an edge $v_r v_{r+2}$ to G so that the resulting graph is still pseudo-outerplanar, a contradiction. Since $v_r v_s$ is a non-crossed chord, $d(v_{r+1}) = 2$. Now remove the vertex v_{r+1} from G and denote the resulting graph by H' . Then by a similar argument as before one can prove that H' is a maximal pseudo-outerplanar diagram. Furthermore, one can easily see that G is obtained from H' by gluing a K_3 along the boundary edge $v_r v_{r+2}$ of H .

Case 2. There exists a non-crossed chord in G .

Let $v_i v_j$ ($1 \leq i < j \leq |G|$) be a non-crossed chord in G . In this case we shall assume that there is no crossed chord in $\mathcal{C}[v_i, v_j]$, because otherwise we are in Case 1. We can also properly choose i and j such that $\mathcal{C}[v_i, v_j] \setminus \{v_i v_j\} = \emptyset$. Therefore, we are now in the second subcase of Case 1, where we can set $r := i$ and $s := j$. \square

Corollary 3.6. *Each pseudo-outerplanar graph can be decomposed into two forests and a matching.*

Proof. Let G be a pseudo-outerplanar graph. In the following, we proceed by induction on the size of G and assume that G is a maximal pseudo-outerplanar diagram. By Theorem 3.5, there respectively exists a $K_3 = [xyz]$ or a $K_4 = [xyuv]$ contained in G such that $H = G - \{xz, yz\}$ or $H = G - \{xu, xv, yu, yv, uv\}$ is a maximal

pseudo-outerplanar graph with xy being its boundary edge. By induction on H , there exists two forests F_1, F_2 and a matching M such that $E(H) = E(F_1) \cup E(F_2) \cup E(M)$. In the former case, let $F'_1 = F_1 + xz$, $F'_2 = F_2 + yz$ and $M' = M$; and in the latter case, let $F'_1 = F_1 + \{xu, xv\}$, $F'_2 = F_2 + \{yu, yv\}$ and $M' = M + uv$. One can easily check that the two forests F'_1, F'_2 and the matching M' are the desired decomposition of G . \square

Theorem 3.7. *For every integer $n \geq 6$, there exists a 2-connected pseudo-outerplanar graph with order n that cannot be decomposed into two forests.*

Proof. Let $C = v_1 \cdots v_n v_1$ ($n \geq 6$) be a cycle with n vertices. We add edges $v_1 v_i$ for all $3 \leq i \leq n-1$ and edges $v_{2i} v_{2i+2}$ for all $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$. One can easily check that the resulted graph G_n is a 2-connected pseudo-outerplanar graph with order n and size $\lfloor \frac{5}{2}n \rfloor - 4$. If G_n can be decomposed into two forests F_1 and F_2 , then $|E(G_n)| = |E(F_1)| + |E(F_2)| \leq |V(F_1)| + |V(F_2)| - 2 \leq 2n - 2$. However, for $n \geq 6$, $|E(G_n)| = \lfloor \frac{5}{2}n \rfloor - 4 > 2n - 2$. Hence, the graph G_n ($n \geq 6$) cannot be covered by two forests. \square

From Corollary 3.6 and Theorem 3.7, we directly have the following two corollaries.

Corollary 3.8. *Every pseudo-outerplanar graph is $(2, 1)$ -coverable; the two parameters given here are best possible.*

Corollary 3.9. *The arboricity of a pseudo-outerplanar graph is at most 3; and this bound is sharp.*

4 Unavoidable Structures

In this section, a vertex set $\mathcal{V}[v_i, v_j]$ ($i < j$) is called a *non-edge* if $j = i + 1$ and $v_i v_j \notin E(G)$, called a *path* if $v_k v_{k+1} \in E(G)$ for all $i \leq k < j$ and called a *subpath* if $j > i + 1$ and some edges in the form $v_k v_{k+1}$ ($i \leq k < j$) are missing. We say a chord $v_k v_l$ ($k < l$) is contained in a chord $v_i v_j$ ($i < j$) if $i \leq k$ and $l \leq j$. In any figure of this section, the solid vertices have no edges of G incident with them other than those shown.

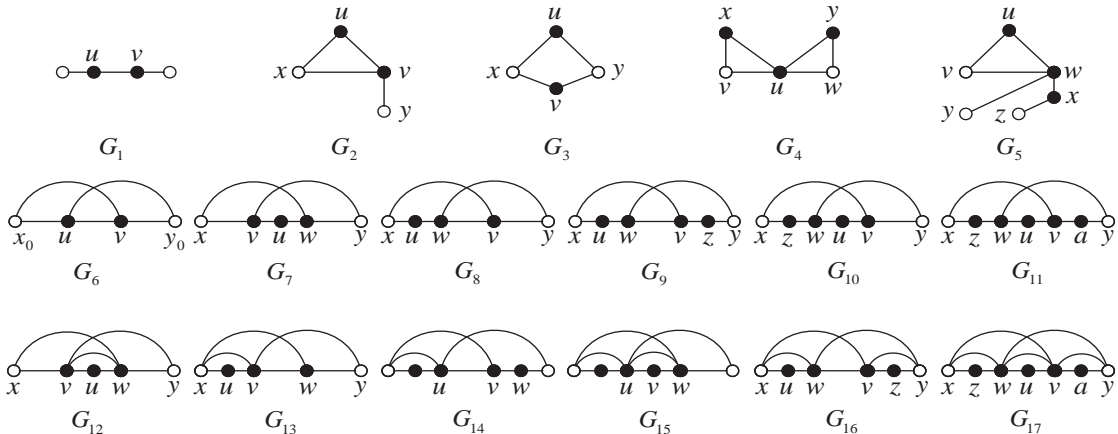
Lemma 4.1. [12] *Let G be a 2-connected outerplanar graph. Then*

- (1) G has two adjacent 2-vertices u and v , or
- (2) G has a 3-cycle wxu such that $d(u) = 2$ and $d(w) = 3$, or
- (3) G has a 4-vertex w , where $N(w) = \{u, v, x, y\}$, such that $d(u) = d(v) = 2$, $N(u) = \{w, x\}$ and $N(v) = \{w, y\}$.

For the class of pseudo-outerplanar graphs, we have a similar structural theorem as Lemma 4.1. But it seems much more complex since crossings are permitted in a pseudo-outerplanar graph.

Theorem 4.2. *Let G be a pseudo-outerplanar diagram with $\delta(G) \geq 2$. Then G contains one of the following configurations G_1 – G_{17} . Moreover,*

- (a) *if G contains some configuration among G_6 – G_{17} , then the drawing of this configuration in the figure is a part of the diagram of G with its bending edges corresponding to the chords;*
- (b) *if G contains the configuration G_3 and $xy \notin E(G)$, where x and y are the vertices of G_3 as described in the figure, then we can properly add an edge xy to G so that the resulting diagram is still pseudo-outerplanar.*



Proof. We first consider the case when G is a 2-connected pseudo-outerplanar diagram. Recall that this diagram minimizes the number of crossings. Let $v_1, v_2, \dots, v_{|G|}$ be the vertices of this diagram lying in a clockwise sequence. If there is no crossings in G , then G is an outerplanar graph and thus G satisfies this theorem by Lemma 4.1. Otherwise, we can properly choose one chord $v_i v_j$ such that

- (1) $v_i v_j$ crosses $v_k v_l$ in G ;
- (2) v_i, v_k, v_j and v_l are lying in a clockwise sequence;
- (3) besides $v_i v_j$ and $v_k v_l$, there is no crossed chords in $\mathcal{C}[v_i, v_l]$.

The condition (3) can be easily fulfilled, because otherwise we could change the values of i and j to meet this condition (note that the values of k and l are determined by i and j). Without loss of generality, assume that $1 \leq i < k < j < l \leq |G|$, because otherwise we can adjust the labellings of the vertices in G to meet it.

Claim 1. $\mathcal{V}[v_i, v_k]$ is either non-edge or path, and so do $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$.

We only need to prove that $\mathcal{V}[v_i, v_k]$ cannot be subpath. Otherwise there exists two vertices v_m and v_{m+1} , where $i \leq m \leq k-1$, such that $v_m v_{m+1} \notin E(G)$. If there are chords in the form $v_a v_{m+1}$ such that $i \leq a \leq m-1$, then we choose one among them such that a is maximum. One can see that v_a is a vertex cut of G , because there is no edges between $\mathcal{V}[v_{a+1}, v_m]$ and $\mathcal{V}[v_{m+1}, v_{a-1}]$ by the choice of a and (3). This contradicts the fact that G is 2-connected. Thus there is no chords in the form $v_a v_{m+1}$ such that $i \leq a \leq m-1$. Similarly, there is no chords in the form $v_m v_b$ such that $m+2 \leq b \leq k$. Let $p = \max\{n | v_{m+1} v_n \in E(G), m+1 < n \leq k\}$ and $q = \min\{n | v_n v_m \in E(G), i \leq n < m\}$. Since $\mathcal{V}[v_i, v_k]$ is neither non-edge nor path, we have $k-i \geq 2$ and thus at least one of the integers p and q exists. Without loss of generality suppose that p exists. Then v_p is a vertex cut of G , because there is no edges between $\mathcal{V}[v_{m+1}, v_{p-1}]$ and $\mathcal{V}[v_{p+1}, v_m]$ by the choices of m, p and by (3). This contradiction completes the proof of Claim 1.

Claim 2. If $\mathcal{V}[v_i, v_k]$ is a path and $k-i \geq 3$, then G has a subgraph isomorphic to one of the configurations $\{G_1, G_2, G_4\}$. This result also holds for $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ if $j-k \geq 3$ and $l-j \geq 3$, respectively.

Suppose that there is no other chord except $v_i v_k$ (if exists) in $\mathcal{V}[v_i, v_k]$, then the configuration G_1 occurs, since $k-i \geq 3$. So we assume that $S := \mathcal{C}[v_i, v_k] \setminus \{v_i v_k\} \neq \emptyset$. Now we prove that there exists at least one chord in S that contains at least one other chord. Suppose that such a chord does not exist. Then we first choose a chord $v_m v_n \in S$ ($m < n$). Without loss of generality, assume that $n \neq k$. If $n-m \geq 3$, then we can easily find a copy of G_1 in G , since $v_m v_n$ contains no other chords by our assumption. If $n-m = 2$, then it is trivial to see that $d(v_{m+1}) = 2$. Now if $\min\{d(v_m), d(v_n)\} \leq 3$, then a copy of G_2 would be found. Thus we shall assume that $\min\{d(v_m), d(v_n)\} \geq 4$. So there exists another chord $v_n v_p$ ($n < p$) in S , since $d(v_n) \geq 4$ and $v_m v_n$ cannot be contained in a chord in the form $v_q v_n$ ($q < n$) by the assumption. Similarly, we shall assume that $p-n = 2$ and $d(v_{n+1}) = 2$ for otherwise the configuration G_1 would be found. Now one can see that $d(v_n) = 4$, because otherwise there would be chord in S that contains either $v_m v_n$ or $v_n v_p$, a contradiction. Therefore, the graph induced by $\mathcal{V}[v_m, v_p]$ contains the configuration G_4 . Thus we can choose one chord $v_a v_b \in S$ ($a < b$) such that $v_a v_b$ contains at least one chord, and furthermore, every chord contained in $v_a v_b$ contains no other chords (this condition can be easily fulfilled by properly changing the values of a and b if necessary). Let $v_m v_n$ ($m < n$) be the chord contained in $v_a v_b$. Then by the similar argument as above, we have to consider the case when $n-m = 2$, $d(v_{m+1}) = 2$ and $\min\{d(v_m), d(v_n)\} \geq 4$. Without loss of generality, assume that $n \neq b$. Then there must be a chord $v_n v_p$ ($n < p \leq b$) in S , since $d(v_n) \geq 4$ and $v_m v_n$ can not be contained in a chord in the form $v_q v_n$ ($q < n$) by the choices of a and b . By the similar argument as before, if G contains no copies of G_1 or G_2 , then $p-n = 2$ and $d(v_{n+1}) = 2$. Furthermore, one can similarly prove that $d(v_n) = 4$ by the choices of a and b . Thus we would find a copy of G_4 in the graph induced by $\mathcal{V}[v_m, v_p]$.

Claim 3. At most one of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ can be non-edge.

If $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_k, v_j]$ are non-edge, then it is trivial that v_l is a vertex cut of G , contradicting the fact that G is 2-connected. If $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_j, v_l]$ are non-edge, then we can adjust the drawing of G by replacing the vertices order $\{v_i, v_k, v_{k+1}, \dots, v_{j-1}, v_j, v_l\}$ with $\{v_i, v_j, v_{j-1}, \dots, v_{k+1}, v_k, v_l\}$. This operation can reduce the number of

crossings in the drawing of G by one, contradicting the assumption that this diagram minimizes the number of crossings.

Claim 4. If one of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ is non-edge, then G has a subgraph isomorphic to one of the configurations $\{G_1, G_2, G_3\}$.

Suppose that $\mathcal{V}[v_i, v_k]$ is a non-edge. By Claims 1–3, both $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ are paths with $1 \leq j - k \leq 2$ and $1 \leq l - j \leq 2$. If $j - k = 2$ and $v_k v_j \in E(G)$, then it is clear that $d(v_k) = 3$ and $d(v_{k+1}) = 2$, implying that the configuration G_2 occurs. If $j - k = 2$ but $v_k v_j \notin E(G)$, then $d(v_k) = d(v_{k+1}) = 2$, implying that the configuration G_1 occurs. So we assume that $j = k + 1$. If $l = j + 2$, then $d(v_{j+1}) = 2$ whenever $v_j v_l$ is an chord or not. In this case the configuration G_3 occurs since $d(v_k) = 2$, and moreover, $G + v_j v_l$ is still pseudo-outerplanar if $v_j v_l \notin E(G)$. So we assume that $l = j + 1$. Now v_k, v_j, v_l form a triangle satisfying $d(v_k) = 2$ and $d(v_j) = 3$. So the configuration G_2 occurs. The case when $\mathcal{V}[v_j, v_l]$ is a non-edge can be dealt with similarly.

Now suppose that $\mathcal{V}[v_k, v_j]$ is a non-edge. By Claims 1–3, both $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_j, v_l]$ are paths with $1 \leq k - i \leq 2$ and $1 \leq l - j \leq 2$. If $k - i = 2$ or $j - l = 2$, by the similar argument as before, we either have $d(v_{k-1}) = d(v_k) = 2$ or have $d(v_j) = d(v_{j+1}) = 2$, implying that the configuration G_1 occurs. So we assume that $k - i = l - j = 1$. In this case the four vertices v_i, v_j, v_l and v_k form a quadrilateral with $d(v_i) = d(v_k) = 2$, which implies that the configuration G_3 occurs in G and furthermore, $G + v_i v_l$ is still pseudo-outerplanar if $v_i v_l \notin E(G)$.

In the following, we assume that $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ are all paths, where $\max\{k - i, j - k, l - j\} \leq 2$. Set $X = \mathcal{C}[v_i, v_l] \setminus \{v_i v_j, v_k v_l\}$ and $x = |X|$. It is clear that $x \leq 3$.

Claim 5. If $x = 0$, then G has a subgraph isomorphic to one of the configurations G_6 – G_{11} ; If $x = 1$, then G has a subgraph isomorphic one of the configurations $\{G_5, G_{12}, G_{13}, G_{14}\}$; If $x = 2$, then G has a subgraph isomorphic to one of the configurations $\{G_5, G_{15}, G_{16}\}$; If $x = 3$, then G has a subgraph isomorphic to the configuration G_{17} .

Here, we just show the case when $x = 2$ and $v_k v_j, v_j v_l \in X$ for example, and leave the discussions about other cases to the readers since they are quite similar. In fact, if $k - i = 1$ (resp. $k - i = 2$), then the configuration G_{15} (resp. G_5) would occurs in G since $d(v_k) = 4$ and $d(v_{i+1}) = d(v_{k+1}) = d(v_{j+1}) = 2$, and furthermore the drawing of the configuration G_{15} (resp. G_5) in the figure is just a part of the diagram of G with its bending edges corresponding to the chords.

Until now, Claims 1–5 just complete the proof of this theorem for the case when G is 2-connected. Now we suppose that G has at least two blocks. Let B be an end block and let $v_1, v_2, \dots, v_{|B|}$ be the vertices of B that lies in a clockwise sequence. Without loss of generality, let v_1 be the unique cut vertex of B .

Claim 6. B is an outerplanar graph.

We prove that there is no crossings in B . Suppose, to the contrary, that there is a chord $v_i v_j$ that crosses another chord $v_k v_l$, where $1 \leq i < k < j < l$. Note that the chord $v_i v_j$ satisfies (1) and (2) now. If it does not fulfill (3) at this stage. Then there must be at least one pair of mutually crossed chords contained in either $\mathcal{C}[v_i, v_k]$, or $\mathcal{C}[v_k, v_j]$, or $\mathcal{C}[v_j, v_l]$. We choose one pair $v_a v_b$ and $v_c v_d$ among them such that $a < c < b < d$ and there is no other crossed chords in $\mathcal{C}[v_a, v_d]$ besides $v_a v_b$ and $v_c v_d$. Now set $i := a$, $j := b$, $k := c$ and $l := d$. Therefore, in any case we can find a pair of mutually crossed chords, $v_i v_j$ and $v_k v_l$, such that $1 \leq i < k < j < l$ and the three conditions at the beginning of the proof are fulfilled. Note that B is an 2-connected pseudo-outerplanar diagram. Thus we can set v_i, v_j, v_k, v_l as we did in the 2-connected case. Recall the proofs of Claims 1–5, every time we find a copy of some configuration the vertices v_i and v_l cannot be the solid vertices (i.e. the degrees of them in the configuration shall not necessarily to be confirmed). For a vertex $v \in V(B) \setminus \{v_1\}$, its degree in B is equal to its degree in G , since B is an end block and v_1 is the unique cut vertex of the B . Among the vertices in $\mathcal{V}[v_i, v_l]$, only v_i may be the cut vertex since $1 \leq i < k < j < l$. Therefore, the proofs of Claims 1–5 are also valid for this claim and then the same results would be obtained.

Claim 7. B has a subgraph isomorphic to one of the configurations $\{G_1, G_2, G_4\}$ in such a way that v_1 is not a solid vertex.

Since B is a 2-connected outerplanar graph, B is hamiltonian. So $\mathcal{V}[v_1, v_{|B|}]$ is a path. The proof of Claim 2

implies that if $\mathcal{V}[v_i, v_k]$ is a path with $k - i \geq 3$ such that there is no crossed edges in $\mathcal{C}[v_i, v_k]$ and no edges between $\mathcal{V}(v_i, v_k)$ and $\mathcal{V}(v_k, v_i)$, then G contains one of $\{G_1, G_2, G_4\}$ in such a way that v_i and v_k are not the solid vertices. Thus in this claim, if $|B| \geq 4$, then we set $i := 1$, $k := |B|$ and come back to the proof of Claim 2. If $|B| \leq 3$, then it is trivial to see that G_1 would appear. This contradiction completes the proof of the theorem for the case when G has cut vertices. \square

The following is a straightforward corollary of Theorem 4.2.

Corollary 4.3. *Each pseudo-outerplanar graph contains a vertex of degree at most 3.*

5 Edge Chromatic Number and Linear Arboricity

In this section, we aim to consider the problems of covering a pseudo-outerplanar graph G with $\Delta(G)$ matchings or $\lceil \frac{\Delta(G)}{2} \rceil$ linear forests. A graph G is χ' -critical if $\chi'(G) = \Delta(G) + 1$ but $\chi'(H) \leq \Delta(G)$ for any proper subgraph $H \subset G$, is la -critical if $la(G) > \lceil \frac{\Delta(G)}{2} \rceil$ but $la(H) \leq \lceil \frac{\Delta(G)}{2} \rceil$ for any proper subgraph $H \subset G$.

Lemma 5.1. *If G is χ' -critical and $uv \in E(G)$, then $d(u) + d(v) \geq \Delta(G) + 2$.*

Lemma 5.2. *If G is la -critical and $uv \in E(G)$, then $d(u) + d(v) \geq 2\lceil \frac{\Delta(G)}{2} \rceil + 2$.*

The above two lemmas are very classic and useful; their proofs can be found in [3] and [14] respectively. Given a coloring φ of G , $c_j(v)$ denotes the number of edges incident with v colored by j . Let $C_\varphi^i(v) = \{j | c_j(v) = i\}$, $i = 0, 1, 2$. Then $C_\varphi^0(v) \cup C_\varphi^1(v) = \{1, 2, \dots, k\}$ if φ is a proper k -edge-coloring, and $C_\varphi^0(v) \cup C_\varphi^1(v) \cup C_\varphi^2(v) = \{1, 2, \dots, k\}$ if φ is a k -tree-coloring. For brevity, in the proof of Theorem 5.3 we use the notion k -coloring to replace the statements of proper k -edge-coloring or k -tree-coloring and use the notion PO -graph to replace the statement of pseudo-outerplanar graph. For a graph G and two distinct vertices $u, v \in V(G)$, denote by $G + xy$ the graph obtained from G by adding a new edge xy if $xy \notin E(G)$, or G itself if $xy \in E(G)$.

Theorem 5.3. *Let G be a pseudo-outerplanar graph. If $\Delta(G) \geq 4$, then $\chi'(G) = \Delta(G)$.*

Proof. Suppose for a contradiction that there exists a minimal (in terms of the size) pseudo-outerplanar diagram G with $\Delta(G) \geq 4$ that has no $\Delta(G)$ -coloring. One can easily observe that G is 2-connected and χ' -critical. By Theorem 4.2 and Lemma 5.1, G contains at least one of the configurations $\{G_3, G_4, G_5, G_6, G_{12}, G_{13}, G_{16}, G_{17}\}$. Set $S = \{1, 2, \dots, \Delta(G)\}$.

If $G \supseteq G_3$, then the pseudo-outerplanar graph $G' = G \setminus \{u, v\}$ admits a $\Delta(G)$ -coloring ϕ by induction hypothesis (when $\Delta(G') = \Delta(G)$) or Vizing's Theorem (when $\Delta(G') \leq \Delta(G) - 1$). Construct a $\Delta(G)$ -coloring φ of G as follows. If $C_\phi^1(x) = C_\phi^1(y) := L$ (notice that $|L| = \Delta(G) - 2$ by Lemma 5.1), then let $\varphi(ux) = \varphi(yv) \in S \setminus L$ and $\varphi(uy) = \varphi(xv) \in S \setminus (L \cup \{\varphi(ux)\})$. If $C_\phi^1(x) \neq C_\phi^1(y)$, then $(S \setminus C_\phi^1(x)) \cap C_\phi^1(y) \neq \emptyset$ since $d(x) = d(y) = \Delta(G)$ by Lemma 5.1. Let $\varphi(ux) \in (S \setminus C_\phi^1(x)) \cap C_\phi^1(y)$, $\varphi(xv) \in S \setminus (C_\phi^1(x) \cup \{\varphi(ux)\})$, $\varphi(yv) \in S \setminus (C_\phi^1(y) \cup \{\varphi(xv)\})$ and $\varphi(uy) \in S \setminus (C_\phi^1(y) \cup \{\varphi(yv)\})$. In each case, we color the remain edges of G by the same colors used in ϕ . Thus, we have constructed a $\Delta(G)$ -coloring φ of G from the $\Delta(G)$ -coloring ϕ of G' . In the next cases, while constructing a coloring φ of G from the coloring ϕ of G' , we only give the colorings for the edges in $E(G) \setminus E(G')$, since for every edge $e \in E(G) \cap E(G')$ we always let $\varphi(e) = \phi(e)$.

If $G \supseteq G_4$, we shall assume that $d(v) = d(w) = \Delta(G) = 4$ because of Lemma 5.1. Then the PO -graph $G' = G \setminus \{x, y, u\}$ admits a 4-coloring ϕ . Construct a 4-coloring φ of G as follows, where two cases are considered without loss of generality (wlog. for short). If $C_\phi^1(v) = C_\phi^1(w) = \{1, 2\}$, then let $\varphi(uy) = 1$, $\varphi(ux) = 2$, $\varphi(uw) = \varphi(vx) = 3$ and $\varphi(uv) = \varphi(wy) = 4$. If $C_\phi^1(v) = \{1, 2\}$, $1 \notin C_\phi^1(w)$ and $3 \in C_\phi^1(w)$, then let $\varphi(uw) = 1$, $\varphi(ux) = 2$, $\varphi(xv) = \varphi(uy) = 3$, $\varphi(uv) = 4$ and $\varphi(wy) \in \{2, 3, 4\} \setminus C_\phi^1(w)$.

If $G \supseteq G_5$, we shall assume that $d(v) = \Delta(G) = 4$ because of Lemma 5.1. Then the PO -graph $G' = G \setminus \{u\}$ admits a 4-coloring ϕ . One can easily see that $(C_\phi^1(v) \cap C_\phi^1(w)) \setminus \{\phi(vw)\} \neq \emptyset$, because otherwise vw would be incident with four colors under ϕ . Assume that $C_\phi^1(v) = \{1, 2, 3\}$ and $\phi(vw) = 3$ wlog. If $C_\phi^1(w) \neq C_\phi^1(v)$, then

assume that $C_\phi^1(w) = \{1, 3, 4\}$ wlog. Whereafter, we can extend ϕ to a 4-coloring φ of G by taking $\varphi(uv) = 4$ and $\varphi(uw) = 2$. If $C_\phi^1(w) = C_\phi^1(v)$, then we consider two subcases. If $\phi(xz) = 4$, then construct a 4-coloring of G by recoloring wx and wv with 3 and 4, and coloring uv and uw with 3 and 2, respectively. If $\phi(xz) \neq 4$, then construct a 4-coloring of G by recoloring wx with 4 and coloring uv and uw with 4 and 2, respectively.

If $G \supseteq G_6$, we shall assume that $\min\{d(x_0), d(y_0)\} \geq 3$ and $\Delta(G) = 4$ by Lemma 5.1. Assume first that $d(x_0) = d(y_0) = 4$. If $x_0y_0 \notin E(G)$, then let $N(x_0) = \{u, v, x_1, x_2\}$ and $N(y_0) = \{u, v, y_1, y_2\}$. Let $G' = G \setminus \{u, v\} + x_0y_0$. By Lemma 4.2, the configuration G_6 is a part of the pseudo-outerplanar diagram of G . Thus G' can also be a PO-graph and thus G' admits a 4-coloring ϕ by the minimality of G . Set $M = \{\phi(x_0x_1), \phi(x_0x_2), \phi(y_0y_1), \phi(y_0y_2)\}$ and $m = |M|$. Since the colors used in ϕ is at most four and $x_0y_0 \in E(G')$, $m \leq 3$ (otherwise the edge x_0y_0 cannot be colored under ϕ because it is already incident with four colored edges). If $m = 3$, assume that $\phi(x_0x_1) = \phi(y_0y_1) = 1$, $\phi(x_0x_2) = 2$ and $\phi(y_0y_2) = 3$ wlog. Now we can extend ϕ to a 4-coloring φ of G by taking $\varphi(uv) = 1$, $\varphi(vy_0) = 2$, $\varphi(ux_0) = 3$ and $\varphi(vx_0) = \varphi(uy_0) = 4$. If $m \leq 2$, assume that $\phi(x_0x_1) = \phi(y_0y_1) = 1$ and $\phi(x_0x_2) = \phi(y_0y_2) = 2$ wlog. Now we can also extend ϕ to a 4-coloring φ of G by taking $\varphi(uv) = 1$, $\varphi(vy_0) = \varphi(ux_0) = 3$ and $\varphi(vx_0) = \varphi(uy_0) = 4$. On the other hand, if $x_0y_0 \in E(G)$, let $N(x_0) = \{u, v, y_0, x_1\}$ and $N(y_0) = \{u, v, x_0, y_1\}$. Then $x_1 \neq y_1$, otherwise by the 2-connectivity of G we have $G \simeq G[\{u, v, x_0, y_0, x_1\}]$, which can be 4-colorable. Consider the graph $G' = G \setminus \{u, v\} - x_0y_0$, which admits a 4-coloring ϕ by the minimality of G . If $\phi(x_0x_1) = \phi(y_0y_1) = 1$, then let $\varphi(uv) = 1$, $\varphi(x_0y_0) = 2$, $\varphi(ux_0) = \varphi(vy_0) = 3$ and $\varphi(vx_0) = \varphi(uy_0) = 4$. If $\phi(x_0x_1) = 1$ and $\phi(y_0y_1) = 2$, then let $\varphi(vy_0) = 1$, $\varphi(ux_0) = 2$, $\varphi(uv) = \varphi(x_0y_0) = 3$ and $\varphi(vx_0) = \varphi(uy_0) = 4$. Second, assume that one of x_0 and y_0 has degree three. Assume that $d(x_0) = 3$ wlog. Let $N(x_0) = \{u, v, w\}$. Consider the PO-graph $G' = G - ux_0$. By the minimality of G , G' has a 4-coloring ϕ . If $A := S \setminus \{\phi(vx_0), \phi(wx_0), \phi(uv), \phi(uy_0)\} \neq \emptyset$ (recall that $S = \{1, 2, 3, 4\}$), then let $\varphi(ux_0) \in A$. Otherwise, assume that $\phi(vx_0) = 1$, $\phi(wx_0) = 2$, $\phi(uv) = 3$ and $\phi(uy_0) = 4$ wlog. Since $d(v) = 3$, $\phi(uy_0) = 4$ and $vy_0 \in E(G')$, v is not incident with the color 4 under ϕ . Thus we can extend ϕ to a 4-coloring of G by recoloring vx_0 with 4 and then coloring ux_0 with 1.

If $G \supseteq G_{12}$, we shall assume that $\Delta(G) = 4$ because of Lemma 5.1. Assume first that $d(x) = d(y) = 4$. If $xy \notin E(G)$, then denote $N(x) = \{v, w, x_1, x_2\}$ and $N(y) = \{v, w, y_1, y_2\}$. Consider the graph $G' = G \setminus \{v, w\} + xy + ux + uy$. Since the configuration G_{12} is a part of the pseudo-outerplanar diagram of G by Lemma 4.2, we can properly add three edges xy , ux and uy to $G \setminus \{v, w\}$ such that G' is still a PO-graph. Thus G' admits a 4-coloring ϕ by the minimality of G . One can see that $\{\phi(xx_1), \phi(xx_2)\} \neq \{\phi(yy_1), \phi(yy_2)\}$ (otherwise we cannot properly color the triangle uxy under ϕ) and $\{\phi(xx_1), \phi(xx_2)\} \cap \{\phi(yy_1), \phi(yy_2)\} \neq \emptyset$ (otherwise we cannot color the edge xy under ϕ). Assume that $\phi(xx_1) = 1$, $\phi(xx_2) = \phi(yy_1) = 2$ and $\phi(yy_2) = 3$ wlog. Then we can construct a 4-coloring φ of G by taking $\varphi(uv) = \varphi(wy) = 1$, $\varphi(vw) = 2$, $\varphi(uw) = \varphi(vx) = 3$ and $\varphi(wx) = \varphi(vy) = 4$. If $xy \in E(G)$, then denote $N(x) = \{v, w, y, x_1\}$ and $N(y) = \{v, w, x, y_1\}$. We shall also assume that $x_1 \neq y_1$ because otherwise $G \simeq G[\{u, v, w, x, y, x_1\}]$ by the 2-connectivity of G , which admits a 4-coloring. Now we remove u, v and w from the diagram of G . Denote by G'' the resulting diagram. Then G'' is a PO-graph so that both x and y has degree two in G'' . Since the diagram of G minimizes the number of crossings, xx_1 does not cross yy_1 in G (and thus in G''). Denote by G' the graph obtained from G'' by contracting the edge xy . From the above arguments, one can see that G' is still a PO-graph with $E(G) \setminus E(G') = \{uv, uw, vw, vx, wx, vy, wy, xy\}$. Furthermore, by the minimality of G , G' admits a 4-coloring ϕ with $\phi(xx_1) \neq \phi(yy_1)$. Suppose that $\phi(xx_1) = 1$ and $\phi(yy_1) = 2$. Then we can construct a 4-coloring φ of G by taking $\varphi(uw) = \varphi(vy) = 1$, $\varphi(uv) = \varphi(wx) = 2$, $\varphi(vw) = \varphi(xy) = 3$ and $\varphi(vx) = \varphi(wy) = 4$. Second, assume that one of x and y , say x wlog., has degree at most three. If $d(x) \leq 2$, then it is easy to see that $G \simeq G[\{u, v, w, x, y\}]$ by the 2-connectivity of G , which admits a 4-coloring. If $d(x) = 3$, then denote $N(x) = \{v, w, x_1\}$. Consider the PO-graph $G' = G - uv$, which admits a 4-coloring ϕ by the minimality of G . If $A := S \setminus \{\phi(uw), \phi(vw), \phi(vy), \phi(vx)\} \neq \emptyset$ (recall that $S = \{1, 2, 3, 4\}$), then let $\varphi(uv) \in A$. Otherwise, assume that $\phi(uw) = 1$, $\phi(vw) = 2$, $\phi(vy) = 3$ and $\phi(vx) = 4$ wlog. It follows that $\phi(wx) = 3$ and $\phi(wy) = 4$. If $\phi(xx_1) = 1$, then we can construct a 4-coloring of G by recoloring vx and uw with 2, recoloring vw with 1 and coloring uv with 4. If $\phi(xx_1) = 2$, then we can again construct a 4-coloring of G by recoloring vx with 1 and coloring uv with 4.

If $G \supseteq G_{13}$, then we shall assume that $d(x) = \Delta(G) = 4$ by Lemma 5.1. Denote the fourth neighbor of x by

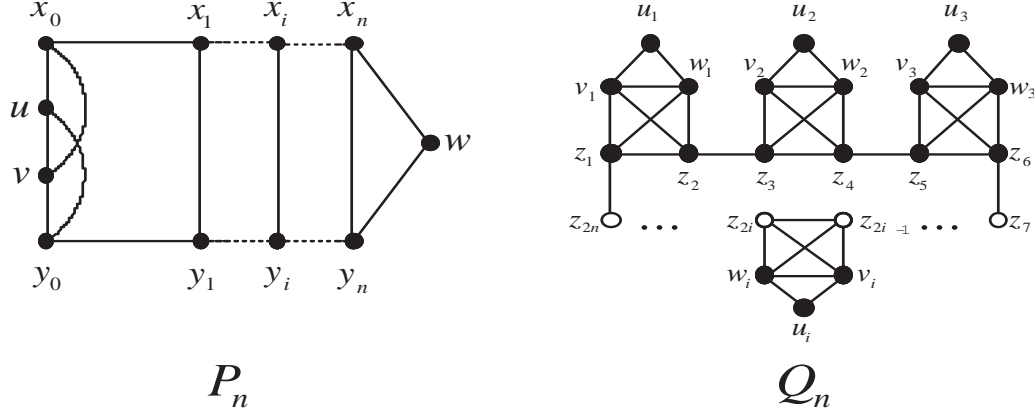


Figure 4: Special pseudo-outerplanar graphs

x_1 and meanwhile assume that $d(y) = 4$ and $N(y) = \{v, w, y_1, y_2\}$ wlog. Then the PO-graph $G' = G \setminus \{u, v, w\}$ admits a 4-coloring ϕ . Wlog. assume that $\phi(xx_1) = 1$. Construct a 4-coloring φ of G as follows. If $1 \in C_\phi^1(y)$ (suppose $\phi(yy_1) = 1$ and $\phi(yy_2) = 2$ wlog.), then let $\varphi(vw) = 1$, $\varphi(uv) = \varphi(wx) = 2$, $\varphi(vx) = \varphi(wy) = 3$ and $\varphi(ux) = \varphi(vy) = 4$. If $1 \notin C_\phi^1(y)$ (suppose $\phi(yy_1) = 2$ and $\phi(yy_2) = 3$ wlog.), then let $\varphi(vy) = 1$, $\varphi(ux) = \varphi(vw) = 2$, $\varphi(uv) = \varphi(wx) = 3$ and $\varphi(vx) = \varphi(wy) = 4$.

If $G \supseteq G_{16}$, then we shall assume that $d(x) = d(y) = \Delta(G) = 4$ by Lemma 5.1. Denote the fourth neighbor of x and y by x_1 and y_1 respectively. Then the PO-graph $G' = G \setminus \{u, v, w, z\}$ admits a 4-coloring ϕ . Construct a 4-coloring φ of G as follows. If $\phi(xx_1) = \phi(yy_1) = 1$, then let $\varphi(vw) = 1$, $\varphi(ux) = \varphi(vz) = \varphi(wy) = 2$, $\varphi(wx) = \varphi(vy) = 3$ and $\varphi(uw) = \varphi(vx) = \varphi(yz) = 4$. If $1 = \phi(xx_1) \neq \phi(yy_1) = 2$, then let $\varphi(vz) = \varphi(wy) = 1$, $\varphi(ux) = \varphi(wy) = \varphi(vz) = 2$, $\varphi(wx) = \varphi(vy) = 3$ and $\varphi(uw) = \varphi(vx) = 4$.

If $G \supseteq G_{17}$, then we shall assume that $d(x) = d(y) = \Delta(G) = 5$ by Lemma 5.1. Then the PO-graph $G' = G \setminus \{u, v, w, z, a\}$ admits a 5-coloring ϕ . Construct a 5-coloring φ of G as follows. If $C_\phi^1(x) = C_\phi^1(y) = \{1, 2\}$, then let $\varphi(uw) = \varphi(av) = 1$, $\varphi(wz) = \varphi(uv) = 2$, $\varphi(xz) = \varphi(vw) = \varphi(ay) = 3$, $\varphi(wx) = \varphi(vy) = 4$ and $\varphi(vx) = \varphi(wy) = 5$. If $|C_\phi^1(x) \cap C_\phi^1(y)| = 1$ (suppose $C_\phi^1(x) = \{1, 2\}$ and $C_\phi^1(y) = \{1, 3\}$ wlog.), then let $\varphi(vw) = 1$, $\varphi(wy) = \varphi(av) = 2$, $\varphi(wz) = \varphi(vx) = 3$, $\varphi(wx) = \varphi(uv) = \varphi(ay) = 4$ and $\varphi(xz) = \varphi(uw) = \varphi(vy) = 5$. If $|C_\phi^1(x) \cap C_\phi^1(y)| = 0$ (suppose $C_\phi^1(x) = \{1, 2\}$ and $C_\phi^1(y) = \{3, 4\}$ wlog.), then let $\varphi(vw) = \varphi(ay) = 1$, $\varphi(wz) = \varphi(vy) = 2$, $\varphi(vx) = \varphi(uw) = 3$, $\varphi(wx) = \varphi(av) = 4$ and $\varphi(xz) = \varphi(uv) = \varphi(wy) = 5$. \square

Theorem 5.4. *For each integer $n \geq 1$, there exists a 2-connected pseudo-outerplanar G with order $2n + 5$ and $\Delta(G) = 3$ so that $\chi'(G) = \Delta(G) + 1$.*

Proof. Let $C = x_0 \cdots x_n w y_n \cdots y_0 v u x_0$ be a cycle. We add edges $x_i y_i$ for all $1 \leq i \leq n$ and add another two edges $x_0 v$ and $y_0 u$ to C . Denote the resulting graph by P_n (See Figure 4). One can easily check that P_n is a 2-connected pseudo-outerplanar graph with $|P_n| = 2n + 5$ and $\Delta(P_n) = 3$. If P_n has a 3-coloring ϕ , then we shall have $\phi(x_0 v) = \phi(y_0 u)$ and $\phi(x_0 u) = \phi(y_0 v)$ (otherwise we cannot color uv properly). Thereby we would deduce that $\phi(x_i x_{i+1}) = \phi(y_i y_{i+1})$ for all $0 \leq i \leq n - 1$ and then $\phi(x_n w) = \phi(y_n w)$. This final contradiction implies that $\chi'(P_n) = \Delta(P_n) + 1 = 4$. \square

Theorem 5.5. *Let G be a pseudo-outerplanar graph. If $\Delta(G) = 3$ or $\Delta(G) \geq 5$, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.*

Proof. Since conjecture 1.1 has already been proved for planar graphs and every PO-graph is planar (cf. Section 1), this theorem holds trivially when $\Delta(G)$ is odd. Thus in the following we assume that $\Delta(G) \geq 6$ and $\Delta(G)$ is even. For brevity we write $k = \frac{\Delta(G)}{2}$. Suppose for a contradiction that there exists a minimal (in terms of the size)

pseudo-outerplanar graph G that has no k -coloring. One can easily observe that G is 2-connected and la-critical. By Theorem 4.2 and Lemma 5.2, G contains the configuration G_3 .

If $xy \notin E(G)$, then by (b) of Lemma 4.2, $G' = G \setminus \{v\} + xy$ is still a PO-graph. Thus by the minimality of G , G' admits a k -coloring ϕ . Now we can construct a k -coloring φ of G by taking $\varphi(vx) = \varphi(vy) = \phi(xy)$ and $\varphi(e) = \phi(e)$ for every $e \in E(G) \cap E(G')$.

If $xy \in E(G)$, then consider the PO-graph $G' = G \setminus \{v\}$, which has a k -coloring ϕ by the minimality of G . It is easy to see that $|C_\phi^1(x)| = |C_\phi^1(y)| = 1$, since $d(x) = d(y) = \Delta(G) = 2k$ by Lemma 5.2. We now construct a coloring φ of G by taking $\varphi(vx) \in C_\phi^1(x)$, $\varphi(vy) \in C_\phi^1(y)$ and $\varphi(e) = \phi(e)$ for every $e \in E(G) \cap E(G')$. If $C_\phi^1(x) \neq C_\phi^1(y)$, then it is easy to see that φ is a k -coloring. If $C_\phi^1(x) = C_\phi^1(y)$, then $\varphi(vx) = \varphi(vy)$ and φ is also a k -coloring unless $\varphi(xy) = \varphi(vx)$ or $\varphi(ux) = \varphi(uy) = \varphi(vx)$. If $\varphi(xy) = \varphi(vx)$, then $\varphi(vx) \notin \{\varphi(ux), \varphi(uy)\}$ and thus we can exchange the colors on ux and vx . One can easily check that the resulting coloring of G is a k -coloring. If $\varphi(ux) = \varphi(uy) = \varphi(vx)$, then we recolor xy with $\varphi(vx)$ and recolor both vx and uy with $\varphi(xy)$. The resulting coloring of G is also a k -coloring. \square

Theorem 5.6. *For each integer $m \geq 1$, there exists a 2-connected pseudo-outerplanar G with order $10m + 5$ and $\Delta(G) = 4$ so that $la(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$.*

Proof. Let $C = z_1 \cdots z_{2n} z_1$ be a cycle and $T_i = u_i v_i w_i u_i$ ($1 \leq i \leq n$) be triangles. Suppose that they are pairwise disjoint. Now for each $1 \leq i \leq n$, add four edges $v_i z_{2i-1}$, $v_i z_{2i}$, $w_i z_{2i-1}$ and $w_i z_{2i}$. Denote the resulting graphs by Q_n (See Figure 4). One can easily check that Q_n is a 2-connected pseudo-outerplanar graph with $\Delta(Q_n) = 4$. Consider the graph Q_{2m+1} ($m \geq 1$). It is trivial that $|Q_{2m+1}| = 10m + 5$ and $la(Q_{2m+1}) \leq 3$ by Lemma 5.2. If Q_{2m+1} has a 2-coloring ϕ , then we shall have $\phi(z_{2i-2} z_{2i-1}) \neq \phi(z_{2i} z_{2i+1})$ for all $1 \leq i \leq 2m + 1$, where $z_0 = z_{4m+2}$ and $z_{4m+3} = z_1$ (otherwise we cannot properly color the set of edges $\{u_i v_i, v_i w_i, w_i u_i, v_i z_{2i-1}, v_i z_{2i}, w_i z_{2i-1}, w_i z_{2i}\}$ for some i). However, the size of the set $\{z_2 z_3, z_4 z_5, \dots, z_{4m+2} z_1\}$ is $2m + 1$, which is odd, but there are only two colors that can be used in ϕ . This final contradiction implies that $la(Q_{2m+1}) = \lceil \frac{\Delta(Q_{2m+1})}{2} \rceil + 1 = 3$. \square

Acknowledgement

The authors thank the referees for many helpful comments and suggestions, which have greatly improved the presentation of the results in this paper, and would also like to acknowledge the editors for pointers to relevant literature and phraseological comments.

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